

SOLUTIONS WITH MULTIPLE ALTERNATE SIGN PEAKS ALONG A BOUNDARY GEODESIC TO A SEMILINEAR DIRICHLET PROBLEM

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ABSTRACT. We study the existence of sign-changing multiple interior spike solutions for the following Dirichlet problem

$$\varepsilon^2 \Delta v - v + f(v) = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega,$$

where Ω is a smooth and bounded domain of \mathbb{R}^N , ε is a small positive parameter, f is a superlinear, subcritical and odd nonlinearity. In particular we prove that if Ω has a plane of symmetry and its intersection with the plane is a two-dimensional strictly convex domain, then, provided that k is even and sufficiently large, a k -peak solution exists with alternate sign peaks aligned along a closed curve near a geodesic of $\partial\Omega$.

1. INTRODUCTION

The present paper is concerned with the following singularly perturbed elliptic problem:

$$\begin{cases} \varepsilon^2 \Delta v - v + |v|^{p-2}v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a smooth and bounded domain of \mathbb{R}^N , $N \geq 2$, $2 < p < \frac{2N}{N-2}$ if $N \geq 3$ and $p > 2$ if $N = 2$, and $\varepsilon > 0$ is a small parameter.

This problem arises from different mathematical models: for instance, it appears in the study of stationary solutions for the Keller-Segel system in chemotaxis and the Gierer-Meinhardt system in biological pattern formation.

In the pioneering paper [22] Ni and Wei proved that for $\varepsilon > 0$ sufficiently small problem (1.1) has a positive least energy solution v_ε which develops a spike layer at the *most centered* part of the domain, i.e. $d_{\partial\Omega}(P_\varepsilon) \rightarrow \max_{P \in \Omega} d_{\partial\Omega}(P)$, where P_ε is the unique maximum of v_ε . Hereafter $d_{\partial\Omega}(P)$ denotes the distance of P from $\partial\Omega$. Since then, there have been many works looking for positive solutions with single and multiple peaks and investigating the location of the asymptotic spikes as well as their profile as $\varepsilon \rightarrow 0^+$. More specifically, several papers study the effect of the geometry of the domain on the existence of positive k -peak solutions (see [3, 4, 7, 8, 10, 11, 12, 14, 15, 17, 20, 25] and references therein). In particular, Dancer and Yan ([10]) proved that if the domain has a nontrivial topology, then there always exists a k -peak positive solution for any $k \geq 1$. This result has been generalized by Dancer, Hillman and Pistoia ([9]) to the case of a not contractible domain. On the other hand, Dancer and Yan ([11]) showed that if Ω is a strictly convex domain and $k \geq 2$, then problem (1.1) does not admit a k -peak positive solutions (see also [24] for the proof when $k = 2$).

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The first result concerning existence of sign changing solutions was obtained by Noussair and Wei ([23]). They proved that for ε sufficiently small (1.1) has a least energy nodal solution with one positive and one negative peak centered at points $P_1^\varepsilon, P_2^\varepsilon$ whose location depends on the geometry of the domain Ω . More precisely, if \bar{P}_1, \bar{P}_2 are the limits of a subsequence of $P_1^\varepsilon, P_2^\varepsilon$, respectively, then (\bar{P}_1, \bar{P}_2) maximizes the function

$$\min \left\{ d_{\partial\Omega}(P_1), d_{\partial\Omega}(P_2), \frac{|P_1 - P_2|}{2} \right\}, \quad P_1, P_2 \in \Omega \times \Omega.$$

Moreover, Wei and Winter ([26]) showed that such solution is odd in one direction when Ω is the unit ball. Successively, Bartsch and Weth in [1, 2], by using a different approach, found a lower bound on the number of sign-changing solutions. These papers are however not concerned with the shape of the solutions.

As far as we know the question of the existence of k -peaked nodal solutions for problem (1.1) for any $k \geq 3$ is largely open. In a general domain, D'Aprile and Pistoia in [13] constructed solutions with h positive peaks and k negative peaks as long as $h + k \leq 6$. They also found solutions with an arbitrarily large number of mixed positive and negative peaks provided some symmetric assumptions are satisfied: in the case of a domain Ω symmetric with respect to a line, where the peaks are aligned with alternate sign along the axis of symmetry, and in the case of a ball, where the peaks are located with alternate sign at the vertices of a regular polygon with an even number of edges.

We believe that it should be possible to extend the above results to a more general domain. More precisely, we conjecture that

- (C1) *there exists a solution with alternate sign peaks aligned on an interior straightline intersecting with $\partial\Omega$ orthogonally;*
- (C2) *there exists a solution with alternate sign peaks aligned on a curve close to a closed geodesic of $\partial\Omega$.*

In the present paper, we prove that the conjecture (C2) is true at least when Ω has a plane of symmetry and its intersection with the plane is a two dimensional strictly convex domain (see the assumptions (a1), (a2) below).

In order to provide the exact formulation of the main result let us fix some notation. We point out that most of the results contained in the aforementioned papers can be extended to equations where $|v|^{p-2}v$ is replaced by a more general nonlinear term. Then we will consider the more general problem

$$\begin{cases} \varepsilon^2 \Delta v - v + f(v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

We will assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is of class $\mathcal{C}^{1+\sigma}$ for some $\sigma > 0$ and satisfies the following conditions:

- (f1) $f(0) = f'(0) = 0$ and $f(t) = -f(-t)$ for any $t \in \mathbb{R}$;
- (f2) $f(t) \rightarrow +\infty$, $f(t) = O(t^{p_1})$, $f'(t) = O(t^{p_2-1})$ as $t \rightarrow +\infty$ for some $p_1, p_2 > 1$ and there exists $p_3 > 1$ such that

$$\forall s, t : \quad |f'(t+s) - f'(t)| \leq \begin{cases} c|s|^{p_3-1} & \text{if } p_3 > 2 \\ c(|s| + |s|^{p_3-1}) & \text{if } p_3 \leq 2 \end{cases}$$

- for a suitable $c > 0$;
 (f3) the following problem

$$\begin{cases} \Delta w - w + f(w) = 0, & w > 0 \quad \text{in } \mathbb{R}^N \\ w(0) = \max_{z \in \mathbb{R}^N} w(z), & \lim_{|z| \rightarrow +\infty} w(z) = 0 \end{cases}$$

has a unique solution w and w is nondegenerate, namely the linearized operator

$$L : H^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N), \quad L[u] := \Delta u - u + f'(w)u,$$

satisfies

$$\text{Kernel}(L) = \text{span} \left\{ \frac{\partial w}{\partial z_1}, \dots, \frac{\partial w}{\partial z_N} \right\}.$$

By the well-known result of Gidas, Ni and Nirenberg ([16]) w is radially symmetric and strictly decreasing in $r = |z|$. Moreover, by classical regularity arguments, the following asymptotic result holds

$$\lim_{|z| \rightarrow +\infty} |z|^{\frac{N-1}{2}} e^{|z|} w(z) = A > 0 \quad \text{and} \quad \lim_{|z| \rightarrow +\infty} \frac{w'(z)}{w(z)} = -1. \quad (1.3)$$

The class of nonlinearities f satisfying (f1)-(f3) includes, and it is not restricted to, the model $f(v) = |v|^{p-2}v$ with $p > 2$ if $N = 1, 2$ and $2 < p < \frac{2N}{N-2}$ if $N \geq 3$. Other nonlinearities can be found in [6].

Here are our assumptions on Ω .

- (a1) Ω is a bounded domain with a C^2 boundary, symmetric with respect to the x_i 's axes for $i = 3, \dots, N$, i.e.

$$(x_1, \dots, x_i, \dots, x_N) \in \Omega \Leftrightarrow (x_1, \dots, -x_i, \dots, x_N) \in \Omega \quad \forall i = 3, \dots, N;$$

- (a2) the relative boundary of $\Omega_0 := \Omega \cap \{x \in \mathbb{R}^N : x_3 = \dots = x_N = 0\}$ has a connected component Γ satisfying

$$\nu_P \cdot (P - Q) > 0 \quad \forall P, Q \in \Gamma, \quad P \neq Q$$

where ν_P is the unit outward normal to $\partial\Omega$ at P .

It is clear that if Ω is a two-dimensional strictly convex domain, the above assumptions are automatically satisfied and, in particular, Γ coincides with the exterior boundary of Ω . More in general, we point out that if $N \geq 3$, then Γ turns out to be a closed geodesic of $\partial\Omega$.

The main purpose of this paper is to prove that if Ω satisfies (a1), (a2) then, provided that δ is sufficiently small and k is even and sufficiently large, the problem (1.2) admits a k -peak solution with k alternate sign peaks aligned near Γ . More precisely, the limiting configuration can be described in the following way: the k peaks lie in Ω_0 and are arranged with alternate sign at distance δ from Γ and the distance between two consecutive peaks is 2δ . Roughly speaking, the limit profile of such solution resembles a crown of peaks surrounding Γ . Moreover the profile of each peak is similar to a translation of the rescaled ground state w . Now we proceed to provide the exact formulation of the result.

Theorem 1.1. *Assume that hypotheses (f1), (f2) and (f3) and (a1), (a2) hold. Then for any $\delta_0 > 0$ there exist $\delta \in (0, \delta_0)$ and an even integer k such that, for ε sufficiently small, the*

problem (1.2) has a solution $v_\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ symmetric with respect to the x_i 's axes for $i = 3, \dots, N$, i.e.

$$v_\varepsilon(x_1, \dots, x_i, \dots, x_N) = v_\varepsilon(x_1, \dots, -x_i, \dots, x_N) \quad \forall i = 3, \dots, N.$$

Furthermore there exist points $P_1^\varepsilon, \dots, P_k^\varepsilon \in \Omega_0$ such that, as $\varepsilon \rightarrow 0^+$,

$$v_\varepsilon(x) = \sum_{i=1}^k (-1)^i w\left(\frac{x - P_i^\varepsilon}{\varepsilon}\right) + o(e^{-\frac{\delta}{2\varepsilon}}) \text{ uniformly for } x \in \overline{\Omega}. \quad (1.4)$$

Moreover, if (P_1^*, \dots, P_k^*) is the limit of a subsequence of $(P_1^\varepsilon, \dots, P_k^\varepsilon)$ as $\varepsilon \rightarrow 0^+$, then¹

$$d_\Gamma(P_i^*) = \delta, \quad P_i^* < P_{i+1}^*, \quad |P_i^* - P_{i+1}^*| = 2\delta \quad \forall i = 1, \dots, k \quad (P_{k+1}^* := P_1^*). \quad (1.5)$$

The assumption that Ω has a plane of symmetry allows to locate the points where the spikes occur along a curve in the plane. Indeed, in the general case the problem of packing the spikes near $\partial\Omega$ in equilibrium is not so simple. We believe that more complicated arrangements should exist depending on the geometry of $\partial\Omega$. In particular, as we mentioned above, we conjecture that a possible balanced pattern may occur for spikes tightly aligned near a closed geodetics of $\partial\Omega$.

We now outline the main idea of the proof of Theorem 1.1.

As with many of the other results mentioned above, a Lyapunov-Schmidt reduction scheme is used in the vicinity of multi-peaked approximate solutions. A sketch of this procedure is given in Section 2. By carrying out the reduction process, we reduce the problem of finding multiple interior spike solutions for (1.2) to the problem of finding critical points of a vector field on the finite dimensional manifold consisting of multi-spike states. More precisely, in order to find such a solution the limiting location of the spikes should be critical for a functional of this type

$$\sum_{i=1}^k e^{-\frac{2d_{\partial\Omega}(P_i) + o(1)}{\varepsilon}} - \sum_{\substack{i,j=1 \\ i < j}}^k (-1)^{i+j} e^{-\frac{|P_i - P_j| + o(1)}{\varepsilon}} + h.o.t. \quad (1.6)$$

on a suitable configuration set in Ω^k . The terms $e^{-\frac{2d_{\partial\Omega}(P_i) + o(1)}{\varepsilon}}$ represent the boundary effect on each spike P_i , created by the boundary condition, while the terms $e^{-\frac{|P_i - P_j| + o(1)}{\varepsilon}}$ are due to the interaction among the peaks which has an attractive or a repulsive effect according to their respective sign. The presence of a factor 2 in the exponentials $e^{-\frac{2d_{\partial\Omega}(P_i) + o(1)}{\varepsilon}}$ suggests that the effect of the boundary acts exactly as an opposite *virtual* peak reflected in $\partial\Omega$. Moreover the setting of Theorem 1.1 suggests that we should restrict ourselves to seeking equilibrium points $P_i \in \Omega_0$, i.e.

$$P_i = (\xi_i, \mathbf{0}), \quad \xi_i \in \mathbb{R}^2, \quad \mathbf{0} = (0, \dots, 0) \in \mathbb{R}^{N-2}.$$

The different interaction effects of the boundary and the peaks, which depend upon their distance in an exponential way, provide the functional described by (1.6) with a suitable local minimum structure.

¹The relation " $P_i^* < P_{i+1}^*$ " refers to a cyclic order on the closed curve $\{P \in \Omega_0 \mid d_\Gamma(P) = \delta\}$.

To give an idea how to apply the minimization argument, let us make the following heuristic considerations. If we look for equilibrium points P_1, \dots, P_k which are packed in a tight strip near Γ in such a way that the neighboring spikes have opposite sign, then the peaks having the same signs are non-interacting to the leading order, and this implies that the terms $e^{-\frac{|P_i - P_j| + o(1)}{\varepsilon}}$ do not contribute to the main term of the expansion if $(-1)^{i+j} = 1$, so (1.6) actually equals

$$\sum_{i=1}^k e^{-\frac{2d_{\partial\Omega}(P_i) + o(1)}{\varepsilon}} + \sum_{\substack{i,j=1 \\ i < j}}^k e^{-\frac{|P_i - P_j| + o(1)}{\varepsilon}} + h.o.t.. \quad (1.7)$$

By (1.7) we get that the spikes P_1, \dots, P_k are repelled from $\partial\Omega$ and doubly from one another, then they may exist in equilibrium when they are packed exactly as in (1.5): indeed the arrangement (1.5) assures that the nonvanishing forces exerted by the boundary and the neighboring spikes balance giving rise to the equilibrium configuration (P_1^*, \dots, P_k^*) .

The paper is organized as follows. Section 2 contains the reduction to the finite dimensional problem, which is done by using the Lyapunov-Schmidt decomposition at the approximate solutions. In Section 3 we study a minimization problem which provides the *equilibrium* arrangement of the alternate sign spikes around Γ ; then we show that the solution of the minimization problem is indeed associated to a solution of (1.2) which satisfies all the properties of Theorem 1.1.

2. THE REDUCTION PROCESS: SKETCH OF THE PROOF

In this section we outline the main steps of the so called *finite dimensional reduction*, which reduces the problem to finding a critical point for a functional on a finite dimensional space. Since this procedure is carried out in a standard way, we omit the proofs and refer to [5, 18, 19, 22] for technical details.

First we introduce some notation and present some important estimates on the approximate solutions. Associated with problem (1.2) is the following energy functional

$$J_\varepsilon(v) = \frac{1}{2} \int_{\Omega} (\varepsilon^2 |\nabla v|^2 + v^2) dx - \int_{\Omega} F(v) dx, \quad v \in H_0^1(\Omega),$$

where $F(t) = \int_0^t f(s) ds$.

For $P \in \Omega$ let $w_{\varepsilon, P}$ be the unique solution of

$$\begin{cases} \varepsilon^2 \Delta v - v + f\left(w\left(\frac{x-P}{\varepsilon}\right)\right) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

$w_{\varepsilon, P}$ is a kind of projection of $w\left(\frac{x-P}{\varepsilon}\right)$ onto the space $H_0^1(\Omega)$.

Then, if we set

$$\psi_{\varepsilon, P}(x) := -\varepsilon \log \left(w\left(\frac{x-P}{\varepsilon}\right) - w_{\varepsilon, P} \right), \quad \psi_\varepsilon(P) = \psi_{\varepsilon, P}(P),$$

it is well known that

$$\psi_{\varepsilon, P}(x) \rightarrow \inf_{z \in \partial\Omega} \{|z - x| + |z - P|\} \quad (2.8)$$

and, consequently,

$$\psi_\varepsilon(P) \rightarrow 2d_{\partial\Omega}(P) \quad (2.9)$$

uniformly for $x \in \overline{\Omega}$ and P on compact subsets of Ω (see [22], for instance).

Fixed $k \geq 1$, we define the configuration space

$$\Lambda_\eta := \left\{ (P_1, \dots, P_k) \in \Omega^k \mid d_{\partial\Omega}(P_i) > \eta \ \forall i, \ |P_i - P_j| > \eta \text{ for } i \neq j \right\}$$

where $\eta > 0$ is a sufficiently small number. For $\mathbf{P} = (P_1, \dots, P_k) \in \Lambda_\eta$ we set

$$w_{\varepsilon, \mathbf{P}} = \sum_{i=1}^k (-1)^i w_{\varepsilon, P_i}.$$

We look for a solution to (1.2) in a small neighborhood of the first approximation $w_{\varepsilon, \mathbf{P}}$, i.e. solutions of the form as $v := w_{\varepsilon, \mathbf{P}} + \phi$, where the rest term ϕ is *small*. To this aim, for $v \in H^2(\Omega)$ we put

$$\mathcal{S}_\varepsilon[v] = \varepsilon^2 \Delta v - v + f(v).$$

Then the problem (1.2) is equivalent to solve

$$\mathcal{S}_\varepsilon[v] = 0, \quad v \in H^2(\Omega) \cap H_0^1(\Omega).$$

We introduce the following approximate cokernel and kernel

$$\begin{aligned} \mathcal{K}_{\varepsilon, \mathbf{P}} &= \text{span} \left\{ \frac{\partial w_{\varepsilon, \mathbf{P}}}{\partial P_i^l} : i = 1, \dots, k, \ l = 1, \dots, N \right\} \subset H^2(\Omega) \cap H_0^1(\Omega), \\ \mathcal{C}_{\varepsilon, \mathbf{P}} &= \text{span} \left\{ \frac{\partial w_{\varepsilon, \mathbf{P}}}{\partial P_i^l} : i = 1, \dots, k, \ l = 1, \dots, N \right\} \subset L^2(\Omega), \end{aligned}$$

denoting by P_i^l the l -th component of P_i for $l = 1, \dots, N$. The idea is that we first solve $\phi = \phi_{\varepsilon, \mathbf{P}}$ in $\mathcal{K}_{\varepsilon, \mathbf{P}}^\perp$, where the orthogonal is taken with respect to the scalar product in $H_0^1(\Omega)$:

$$\langle u, v \rangle_\varepsilon = \int_\Omega (\varepsilon^2 \nabla u \nabla v + uv) \, dx.$$

The following lemma is proved in [5, 19].

Lemma 2.1. *Provided that $\varepsilon > 0$ is sufficiently small, for every $\mathbf{P} \in \Lambda_\eta$ there exists a unique $\phi_{\varepsilon, \mathbf{P}} \in \mathcal{K}_{\varepsilon, \mathbf{P}}^\perp$ such that*

$$\mathcal{S}_\varepsilon[w_{\varepsilon, \mathbf{P}} + \phi_{\varepsilon, \mathbf{P}}] \in \mathcal{C}_{\varepsilon, \mathbf{P}}. \quad (2.10)$$

Moreover the map $\mathbf{P} \in \Lambda_\eta \mapsto \phi_{\varepsilon, \mathbf{P}} \in H_0^1(\Omega)$ is \mathcal{C}^1 and

$$|\phi_{\varepsilon, \mathbf{P}}| \leq C \varepsilon^{-(1+\frac{\sigma}{2})} \frac{\varphi_k(\mathbf{P})}{\varepsilon} \quad (2.11)$$

where the function $\varphi_k : \Omega^k \rightarrow \mathbb{R}$ is defined by

$$\varphi_k(\mathbf{P}) := \min_{\substack{i, j=1, \dots, k \\ i \neq j}} \left\{ d_\Gamma(P_i), \frac{|P_i - P_j|}{2} \right\}, \quad \mathbf{P} := (P_1, \dots, P_k).$$

After that, we define a new functional:

$$M_\varepsilon : \Lambda_\eta \rightarrow \mathbb{R}, \quad M_\varepsilon[\mathbf{P}] := \frac{\varepsilon^{-N}}{\gamma} J_\varepsilon[w_{\varepsilon, \mathbf{P}} + \phi_{\varepsilon, \mathbf{P}}] - \frac{c_1}{\gamma}$$

where $\phi_{\varepsilon, \mathbf{P}}$ has been constructed in Lemma 2.1 and

$$c_1 = \frac{k}{2} \int_{\mathbb{R}^3} |\nabla w|^2 dx - k \int_{\mathbb{R}^3} F(w) dx, \quad \gamma = \int_{\mathbb{R}^3} f(w) e^{x_1} dx.$$

Next proposition contains the key expansion of M_ε (see [19] for the proof).

Proposition 2.2. *The following asymptotic expansions hold:*

$$M_\varepsilon[\mathbf{P}] = \frac{1}{2} (1 + o(1)) \sum_{i=1}^k e^{-\frac{\psi_\varepsilon(P_i)}{\varepsilon}} - (1 + o(1)) \sum_{\substack{i,j=1 \\ i < j}}^k (-1)^{i+j} w\left(\frac{P_i - P_j}{\varepsilon}\right) \quad (2.12)$$

uniformly for $\mathbf{P} = (P_1, \dots, P_k) \in \Lambda_\eta$.

Remark 2.3. *By using (2.9) and (1.3) the expansion (2.12) can be rewritten as*

$$M_\varepsilon[\mathbf{P}] = \sum_{i=1}^k e^{-\frac{2d_{\partial\Omega}(P_i) + o(1)}{\varepsilon}} - \sum_{\substack{i,j=1 \\ i < j}}^k (-1)^{i+j} e^{-\frac{|P_i - P_j| + o(1)}{\varepsilon}}$$

uniformly for $\mathbf{P} = (P_1, \dots, P_k) \in \Lambda_\eta$.

Finally the next lemma concerns the relation between the critical points of M_ε and those of J_ε . It is quite standard in singular perturbation theory; its proof can be found in [19], for instance.

Lemma 2.4. *Let $\mathbf{P}_\varepsilon \in \Lambda_\eta$ be a critical point of M_ε . Then, provided that $\varepsilon > 0$ is sufficiently small, the corresponding function $v_\varepsilon = w_{\varepsilon, \mathbf{P}_\varepsilon} + \phi_{\varepsilon, \mathbf{P}_\varepsilon}$ is a solution of (1.2).*

We finish this section with a symmetry property of the reduction process.

Lemma 2.5. *Suppose \emptyset is invariant under the action of an orthogonal transformation $T \in \mathcal{O}(N)$. Let $\Lambda_\eta^T := \{\mathbf{P} \in \Lambda_\eta : TP_i = P_i \ \forall i\}$ denote the fixed point set of T in Λ_η . Then a point $\mathbf{P} \in \Lambda_\eta^T$ is a critical point of J_ε if it is a critical point of the constrained functional $J_\varepsilon|_{\Lambda_\eta^T}$.*

Proof. We first investigate the symmetry inherited by the function $\phi_{\varepsilon, \mathbf{P}}$ obtained in Lemma 2.1. Setting $T\mathbf{P} := (TP_1, \dots, TP_k)$ for $\mathbf{P} = (P_1, \dots, P_k) \in \emptyset^k$, we claim that

$$\phi_{\varepsilon, \mathbf{P}} = \phi_{\varepsilon, T\mathbf{P}} \circ T \quad \forall \mathbf{P} \in \Lambda_\eta. \quad (2.13)$$

Indeed, because of the symmetry of the domain, we see that

$$w_{\varepsilon, P_i} = w_{\varepsilon, TP_i} \circ T$$

and

$$\mathcal{K}_{\varepsilon, \mathbf{P}} = \{f \circ T \mid f \in \mathcal{K}_{\varepsilon, T\mathbf{P}}\}, \quad \mathcal{K}_{\varepsilon, \mathbf{P}}^\perp = \{f \circ T \mid f \in \mathcal{K}_{\varepsilon, T\mathbf{P}}^\perp\}.$$

Then the function $\phi_{\varepsilon, T\mathbf{P}} \circ T$ belongs to $\mathcal{K}_{\varepsilon, \mathbf{P}}^\perp$ and satisfies (2.10) and (2.11). The uniqueness of the solution ϕ implies (2.13). Therefore the functional J_ε satisfies

$$J_\varepsilon(\mathbf{P}) = J_\varepsilon(T\mathbf{P}).$$

The lemma follows immediately. \square

3. A MINIMIZATION PROBLEM

In this section we will employ the reduction approach to construct the solutions stated in Theorem 1.1. The results obtained in the previous section imply that our problem reduces to the study of critical points of the functional M_ε . In what follows, we assume assumptions (a1), (a2). We get the following result.

Lemma 3.1. *If \mathbf{P} is a critical point of $M_\varepsilon|_{\Omega_0}$, then \mathbf{P} is a critical point of M_ε .*

Proof. This is an immediate consequence of Lemma 2.5. \square

From Lemma 3.1, we need to find a critical point of the functional $M_\varepsilon|_{\Omega_0}$.

Now we set up a maximization problem for the function φ_k defined by

$$\varphi_k(\mathbf{P}) := \min_{\substack{i,j=1,\dots,k \\ i \neq j}} \left\{ d_{\partial\Omega}(P_i), \frac{|P_i - P_j|}{2} \right\}, \quad \mathbf{P} := (P_1, \dots, P_k).$$

The function φ_k appears naturally in the location of the asymptotic spikes, as we will see at the end of the proof of Theorem 1.1.

First we need some auxiliary lemmas.

Lemma 3.2. *Let Γ be as in (a2). For any $\delta_0 > 0$ there exist $\delta \in (0, \delta_0)$ and an even integer k such that*

$$\sup \{ \varphi_k(\mathbf{P}) \mid P_i \in \gamma_\delta, P_i < P_{i+1} \ (P_{k+1} := P_1) \} = \delta$$

where

$$\gamma_\delta := \{P \in \Omega_0 \mid d_\Gamma(P) = \delta\}.$$

Moreover, if $\mathbf{P}^* = (P_1^*, \dots, P_k^*) \in (\gamma_\delta)^k$ is such that $\varphi_k(\mathbf{P}^*) = \delta$, then the points P_1^*, \dots, P_k^* satisfy (1.5), i.e. they form a polygonal having vertices on γ_δ and edge 2δ .

Proof. The strict convexity of Γ implies that, if $\delta_0 > 0$ is sufficiently small, then for any $\delta \in (0, \delta_0]$ we have that γ_δ is a regular closed curve and

$$\text{every point of } \gamma_\delta \text{ has exactly two points on } \gamma_\delta \text{ at distance } 2\delta. \quad (3.14)$$

Then choose $k \in \mathbb{N}$ such that k is even and satisfy

$$k > \frac{\ell(\Gamma)}{2\delta_0},$$

where $\ell(\Gamma)$ denotes the length of the curve Γ . We define

$$\Sigma = \left\{ \delta \in (0, \delta_0] \mid \exists P_1, \dots, P_k \in \gamma_\delta \text{ s.t. } P_i < P_{i+1}, \ |P_i - P_{i+1}| \geq 2\delta \right\}.$$

The definition of k implies that $\delta_0 \notin \Sigma$. On the other hand it is easy to prove that Σ contains δ if $\delta \ll \delta_0$. Let us define δ^* as

$$\delta^* = \sup\{\delta \mid \delta \in \Sigma\}.$$

A straightforward computation shows that δ^* is actually a maximum (it is sufficient to consider a maximizing sequence and then pass to the limit for a convergent subsequence), hence $\delta^* \in (0, \delta_0)$. We point out that $d_{\partial\Omega}(P) = d_\Gamma(P) = \delta$ for any $P \in \gamma_\delta$ provided that δ is small enough, so we clearly have

$$\sup_{\mathbf{P} \in (\gamma_{\delta^*})^k} \varphi_k(\mathbf{P}) = \delta^*.$$

Let $\mathbf{P}^* = (P_1^*, \dots, P_k^*) \in (\gamma_{\delta^*})^k$ be such that $\varphi_k(\mathbf{P}^*) = \delta^*$, which implies

$$\min_{i \neq j} |P_i^* - P_j^*| \geq 2\delta^*.$$

We claim that

$$|P_{i+1}^* - P_i^*| = 2\delta^* \quad \forall i = 1, \dots, k. \quad (3.15)$$

Indeed, assume by contradiction that $|P_2^* - P_1^*| > 2\delta^*$. Then, using (3.14), we can move the points P_i^* 's slightly backwards into new points P_i 's:

$$P_1 = P_1^*, \quad P_i < P_i^* < P_{i+1} \quad \forall i = 2, \dots, k,$$

and the P_i 's verify

$$|P_{i+1} - P_i| > 2\delta^* \quad \forall i = 1, \dots, k.$$

Consider \bar{P}_i the projection of P_i onto γ_δ : by continuity, if $\delta > \delta^*$ is sufficiently closed to δ^* , the \bar{P}_i 's satisfy

$$|\bar{P}_{i+1} - \bar{P}_i| > 2\delta \quad \forall i = 1, \dots, k$$

which contradicts the maximality of δ^* .

Then (3.15) holds, which implies that the P_i^* 's satisfy (1.5). □

Lemma 3.3. *Let $D \subset \mathbb{R}^N$ be a strictly convex domain. Then, for any $\delta > 0$ there exists $\eta > 0$ such that, if $P, Q \in \partial D$, $|P - Q| \geq \delta$ and $\eta_1, \eta_2 \in [0, \eta]$, $(\eta_1, \eta_2) \neq (0, 0)$, then*

$$|P - \nu_P \eta_1 - Q + \nu_Q \eta_2| < |P - Q|,$$

where ν_P is the unit outward normal to ∂D at P .

Proof. Fixed $\delta > 0$, by the strict convexity we get

$$\inf_{P \in \partial D, |Q - P| \geq \delta} \nu_P \cdot (P - Q) = \eta > 0.$$

For $|P - Q| \geq \delta$, $\eta_1, \eta_2 \in [0, \eta]$ with $(\eta_1, \eta_2) \neq (0, 0)$, we compute

$$\begin{aligned} |P - \nu_P \eta_1 - Q + \nu_Q \eta_2|^2 - |P - Q|^2 &= 2\eta_1(Q - P) \cdot \nu_P + 2\eta_2(P - Q) \cdot \nu_Q \\ &\quad + \eta_1^2 + \eta_2^2 - 2\eta_1\eta_2\nu_P \nu_Q \\ &\leq -2(\eta_1 + \eta_2)\eta + (\eta_1 + \eta_2)^2 < 0. \end{aligned}$$

□

With the help of the previous two lemmas we can now give the following result which will be crucial for the asymptotic locations of the k spikes in the solutions of Theorem 1.1.

Proposition 3.4. *Assume that Ω satisfies (a1)-(a2). For any $\delta_0 > 0$ there exist $\delta \in (0, \delta_0)$ and an even integer k such that, if η is sufficiently small, then*

$$\sup_{\mathbf{P} \in \partial U_\eta} \varphi_k(\mathbf{P}) < \sup_{\mathbf{P} \in U_\eta} \varphi_k(\mathbf{P}) = \delta \quad (3.16)$$

where

$$U_\eta = \{\mathbf{P} \in \Omega_0^k \mid \delta - \eta < d_\Gamma(P_i) < \delta + \eta, \bar{P}_i < \bar{P}_{i+1} \ \forall i, \ |P_i - P_j| > 2\delta - \eta \text{ for } i \neq j\}.$$

Here \bar{P} denotes the projection of P onto the curve $\gamma_\delta := \{P \in \Omega_0 \mid d_\Gamma(P) = \delta\}$.

Moreover if $\mathbf{P}^* = (P_1^*, \dots, P_k^*) \in U_\eta$ is such that $\varphi_k(\mathbf{P}^*) = \delta$, then the points P_1^*, \dots, P_k^* satisfy (1.5).

Proof. Let $\delta \in (0, \delta_0)$ and $k \in \mathbb{N}$ even be such that Lemma 3.2 holds. Let D be the strictly convex bounded flat domain whose boundary is γ_δ , which is contained in Ω_0 . According to Lemma 3.3, if $\eta \in (0, \frac{\delta}{2})$ is sufficiently small, then, for any $Q, Q' \in \gamma_\delta$ with $|Q - Q'| \geq \frac{\delta}{2}$ and any $\eta_1, \eta_2 \in [0, \eta]$, $(\eta_1, \eta_2) \neq (0, 0)$, we get

$$|Q - \eta_1 \nu_Q - Q' + \eta_2 \nu_{Q'}| < |Q - Q'|. \quad (3.17)$$

We are going to prove that, for such η ,

$$\mathbf{P} \in \partial U_\eta \implies \varphi_k(\mathbf{P}) < \delta. \quad (3.18)$$

It is useful to point out that for any $P \in \partial U_\eta$ we have $d_\Gamma(P) = d_{\partial\Omega}(P)$, provided δ is small enough. Then it is immediate that, if $d_\Gamma(P_i) < \delta$ for some i or $|P_i - P_j| < 2\delta$ for some $i \neq j$, then $\varphi_k(\mathbf{P}) < \delta$. Moreover, if $\bar{P}_i = \bar{P}_{i+1}$ for some i , then, by construction $|P_i - P_{i+1}| \leq 2\eta < \delta$, and again we get $\varphi_k(\mathbf{P}) < \delta$. Therefore, without loss of generality we may assume

$$d_\Gamma(P_i) \geq \delta, \quad \bar{P}_i < \bar{P}_{i+1} \ \forall i, \quad d_\Gamma(P_1) = \delta + \eta, \quad |P_i - P_j| \geq 2\delta \ \forall i \neq j.$$

Consider \bar{P}_i the projections of P_i onto γ_δ , i.e.

$$P_i = \bar{P}_i - \eta_i \nu_{\bar{P}_i}, \quad \bar{P}_i \in \gamma_\delta, \quad \eta_i \in [0, \eta].$$

If there exist $i \neq j$ such that $|\bar{P}_i - \bar{P}_j| \leq \frac{\delta}{2}$, then,

$$\begin{aligned} |P_i - P_j|^2 &= |\bar{P}_i - \eta_i \nu_{\bar{P}_i} - \bar{P}_j + \eta_j \nu_{\bar{P}_j}|^2 \\ &= |\bar{P}_i - \bar{P}_j|^2 + |\eta_i \nu_{\bar{P}_i} - \eta_j \nu_{\bar{P}_j}|^2 - 2\langle \bar{P}_i - \bar{P}_j, \eta_i \nu_{\bar{P}_i} - \eta_j \nu_{\bar{P}_j} \rangle \\ &\leq |\bar{P}_i - \bar{P}_j|^2 + 4\eta |\bar{P}_i - \bar{P}_j| + 4\eta^2 \leq \frac{\delta^2}{4} + 2\delta\eta + 4\eta^2 < 4\delta^2, \end{aligned}$$

and so $\varphi_k(\mathbf{P}) < \delta$, by which (3.18) follows. Now assume $|\bar{P}_i - \bar{P}_j| \geq \frac{\delta}{2}$ if $i \neq j$. Then $|\bar{P}_i - \bar{P}_j| \geq |P_i - P_j|$ by (3.17). If $|\bar{P}_i - \bar{P}_j| < 2\delta$ for some $i \neq j$, then again $\varphi_k(\mathbf{P}) < \delta$ and we have done. Now assume $|\bar{P}_i - \bar{P}_j| \geq 2\delta$ for every $i \neq j$, which means $\varphi_k(\mathbf{P}) = \delta$. By Lemma 3.2 $|\bar{P}_2 - \bar{P}_1| = 2\delta$. Then (3.17) implies $|P_2 - P_1| < |\bar{P}_2 - \bar{P}_1| = 2\delta$, by which $\varphi_k(\mathbf{P}) < \delta$, and (3.18) follows. Combining (3.18) with Lemma 3.2 we obtain the thesis. \square

Proof of Theorems 1.1 completed. Let us fix $\delta_0 > 0$ is sufficiently small such that for any $\delta \in (0, \delta_0]$ γ_δ is a regular closed curve and satisfies (3.14). Then, let us take $\delta \in (0, \delta_0)$, $k \in \mathbb{N}$ even such that Proposition 3.4 holds for $\eta > 0$ sufficiently small. By (3.14) we deduce

$$\min \left\{ \frac{1}{2} \min_{\substack{j-i \geq 2 \\ (i,j) \neq (1,k)}} |P_i - P_j| \left| P_i \in \gamma_\delta, P_i < P_{i+1} \quad \forall i, |P_i - P_j| \geq 2\delta \quad \forall i \neq j \right. \right\} > \mu > \delta$$

for some suitable $\mu > 0$. Hence, possibly reducing the number η , we may assume

$$\min \left\{ \frac{1}{2} \min_{\substack{j-i \geq 2 \\ (i,j) \neq (1,k)}} |P_i - P_j| \left| \mathbf{P} = (P_1, \dots, P_k) \in U_\eta \right. \right\} > \mu > \delta.$$

Now, if $\mathbf{P} = (P_1, \dots, P_k) \in U_\eta$ and $i < j$ is such that $(-1)^{i+j} = 1$, then $j - i \geq 2$ and, recalling also that k is even, $(i, j) \neq (1, k)$, consequently, $|P_i - P_j| \geq \mu$. By using Remark 2.3 we get

$$\begin{aligned} M_\varepsilon[\mathbf{P}] &= \sum_{i=1}^k e^{-\frac{2d_{\partial\Omega}(P_i)+o(1)}{\varepsilon}} + \sum_{\substack{i,j=1 \\ i < j}}^k e^{-\frac{|P_i - P_j|+o(1)}{\varepsilon}} + O(e^{-\frac{\mu}{\varepsilon}}) \\ &= e^{-\frac{2\varphi_k(\mathbf{P})+o(1)}{\varepsilon}} + O(e^{-\frac{\mu}{\varepsilon}}) \end{aligned}$$

uniformly for $\mathbf{P} = (P_1, \dots, P_k) \in U_\eta$. Proposition 3.4 applies and gives

$$M_\varepsilon[\mathbf{P}^*] = e^{-\frac{2\delta+o(1)}{\varepsilon}}$$

where $\mathbf{P}^* \in U_\eta$ is such that $\varphi_k(\mathbf{P}^*) = \delta$, and

$$\inf_{\mathbf{P} \in \partial U_\eta} M_\varepsilon[\mathbf{P}] \geq e^{-\frac{2\delta'}{\varepsilon}}$$

for some $\delta' < \delta$. We conclude that M_ε has a minimum point $\mathbf{P}_\varepsilon = (P_1^\varepsilon, \dots, P_k^\varepsilon)$ in U_η . According to Lemma 2.4 and Lemma 3.1, for $\varepsilon > 0$ sufficiently small $v_\varepsilon := w_{\varepsilon, \mathbf{P}_\varepsilon} + \phi_{\varepsilon, \mathbf{P}_\varepsilon}$ solves the problem (1.2). Finally if (P_1^*, \dots, P_k^*) is the limit of a subsequence of $(P_1^\varepsilon, \dots, P_k^\varepsilon)$, Proposition 3.4 implies that (P_1^*, \dots, P_k^*) satisfies (1.5).

Thus the thesis of Theorem 1.1 holds, (1.4) following from (2.8) and Lemma 2.1.

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